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Conservation laws of the Camassa–Holm equation

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Abstract
We use the bi-Hamiltonian structure of the Camassa–Holm equation to show that its conservation laws $H_n[m]$ are homogeneous with respect to the scaling $m \mapsto \lambda m$. Moreover, a direct argument is presented proving that $H_{-1}, H_{-2}, \ldots,$ are of local character. Finally, simple representations of the conservation laws in terms of their variational derivatives are derived and used to obtain a constructive scheme for computation of the H_ns.

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1. Introduction

The nonlinear dispersive equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

first arose in 1981 as an abstract equation admitting a bi-Hamiltonian structure [13]. It was rediscovered a decade later by Camassa and Holm [2] as a model for the unidirectional propagation of shallow water waves over a flat bottom, $u(x, t)$ representing the water's free surface in non-dimensional variables. Subsequently, equation (1.1) was obtained independently as a model for nonlinear waves in cylindrical hyperelastic rods [12]. Equation (1.1) is a re-expression of the geodesic flow in the group of compressible diffeomorphisms of the circle [19, 9, 10] and an infinite-dimensional completely integrable Hamiltonian system [1, 3, 5, 11, 16]. The equation admits, in addition to smooth waves, a multitude of travelling wave solutions with singularities—peakons, cuspons, stumpons and composite waves [2, 17, 18]. It has solutions that are global in time [5] as well as solutions modelling wave breaking [4, 7, 8]. Associated with (1.1) there is a whole hierarchy of integrable equations [15].

Introducing the momentum $m = u - u_{xx}$, equation (1.1) can be expressed as a bi-Hamiltonian system [2]

$$m_t = \mathcal{E}\delta H_1[m] = \mathcal{D}\delta H_2[m],$$

with Hamiltonians

$$H_1[m] = \frac{1}{2}\int mu\,\mathrm{d}x, \qquad H_2[m] = \frac{1}{2}\int \left(u^3 + uu_x^2\right)\mathrm{d}x,$$

and corresponding operators

$$\mathcal{D} = -\left(D_x - D_x^3\right), \qquad \mathcal{E} = -(mD_x + D_x m).$$

Accordingly, a recursive argument gives rise to an infinite sequence of quantities

$$\ldots, H_{-2}[m], H_{-1}[m], H_0[m], H_1[m], H_2[m], H_3[m], \ldots,$$

conserved under the flow of (1.1).

As the expressions for the H_ns rapidly get very involved, except for the simplest cases, their structure is largely unknown. One question that has attracted interested is the local or non-local nature of the functionals.

Considering an 'associated Camassa–Holm equation', it was shown in [14] that equation (1.1) admits an infinite sequence of both local and non-local conservation laws. It was conjectured that the conservation laws $H_{-1}, H_{-2}, \ldots$ are all local. By a different method (showing that (1.1) describes pseudo-spherical surfaces and expanding the corresponding quadratic pseudo-potential in power series), this conjecture was proved in [21].

In this paper, we give a direct argument using the bi-Hamiltonian structure to show that $H_{-1}, H_{-2}, \ldots$ are of local character.

Furthermore, we show that all the H_ns are homogeneous in the sense that

$$H_n[\lambda m] = \lambda^{n+1} H_n[m], \qquad \lambda \in \mathbb{R}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

and

$$H_{-n}[\lambda m] = \lambda^{3/2-n} H_{-n}[m], \qquad \lambda \in \mathbb{R}, \quad n = 1, 2, \ldots. \tag{1.3}$$

We also establish the interesting identities

$$H_n[m] = \frac{1}{n+1}\int m\delta H_n[m]\,\mathrm{d}x, \qquad n = 0, 1, 2, \ldots, \tag{1.4}$$

and

$$H_{-n}[m] = \frac{1}{3/2-n}\int m\delta H_{-n}[m]\,\mathrm{d}x, \qquad n = 1, 2, \ldots. \tag{1.5}$$

where $\delta H_n[m]$ denotes the variational derivative of H_n. Using these formulae, we obtain a constructive scheme for computing the H_ns. As an application, we derive explicit expressions for the conservation laws H_{-3} and H_{-4}.

Section 2 reviews some notation and definitions. In section 3, we consider conservation laws of (1.1). Lowering and raising of the H_ns is studied in sections 4 and 5, respectively. Finally, in section 6 we present the algorithm for computation of the conservation laws.

2. Preliminaries

2.1. Functionals

For an integer $n \geqslant 1$, we let H^n be the Sobolev space of all square integrable functions $f \in L^2$ with distributional derivatives $\partial_x^i f \in L^2$ for $i = 1, \ldots, n$.

We will work in some function space $X \subset L^2$. Typically X is some Sobolev space or the Schwartz space of rapidly decreasing functions. X could consist of either periodic functions or functions on the real line with sufficient decay at infinity—what is important is that the boundary terms vanish when integrating by parts. We let x be the independent variable and $m, v \in X$ be functions. Total differentiation with respect to x will be denoted by D_x.

Let $F : X \to \mathbb{R}$ be a functional. Suppose the directional derivative

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F[m+\epsilon v], \qquad m, v \in X$$

defines a continuous linear functional of $v \in X$ for every fixed $m \in X$. If this linear functional can be expressed as a scalar product inherited from L^2,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F[m+\epsilon v] = \int \delta F[m] \cdot v \,\mathrm{d}x, \qquad m, v \in X,$$

we call $\delta F[m]$ the variational derivative of F at m.

A *differential function* P is a smooth function of x, m and derivatives of m up to some finite order. We write

$$P[m] = P(x, m, m_x, m_{xx}, \ldots).$$

Note that the value of $P[m]$ at x may depend only on the value of m and its derivatives evaluated at the point x. A *local functional* is a mapping of the form

$$m \mapsto \int P[m] \,\mathrm{d}x,$$

for some differential function $P[m]$.

Let $P[m]$ be a differential function. The *Fréchet derivative* of P is the differential operator D_P defined by

$$D_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} P[m+\epsilon Q[m]],$$

for any differential function $Q[m]$. The Fréchet derivative of a differential function always exists (see [20]).

The *Euler operator* is given by

$$\mathbf{E} = \frac{\partial}{\partial m} - D_x \frac{\partial}{\partial m_x} + D_x^2 \frac{\partial}{\partial m_{xx}} - D_x^3 \frac{\partial}{\partial m_{xxx}} + \cdots. \tag{2.1}$$

For a local functional $F[m] = \int L[m] \,\mathrm{d}x$ we have the basic equality

$$\delta F[m] = \mathbf{E}(L).$$

2.2. Hamiltonian structure

A linear operator $\mathcal{D}$ on X is *Hamiltonian* if its bi-linear *Poisson bracket* defined by

$$\{F, H\} = \int \delta F \cdot \mathcal{D}\delta H \,\mathrm{d}x,$$

is skew-symmetric

$$\{F, H\} = -\{H, F\}$$

and satisfies the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

Note that the Poisson bracket of two functionals is a new functional

$$\{F, H\}[m] = \int \delta F[m] \cdot \mathcal{D}\delta H[m] \,\mathrm{d}x.$$

Two Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ are *compatible* if their sum $\mathcal{D}+\mathcal{E}$ is still a Hamiltonian operator. A differential operator $\mathcal{D}$ is *nondegenerate* if there is no nonzero differential operator $\tilde{\mathcal{D}}$ such that $\tilde{\mathcal{D}} \cdot \mathcal{D} \equiv 0$. In the scalar case, every differential operator is nondegenerate (see [20]). The following lemma will prove useful when we construct conservation laws.

Lemma 1 (lemma 7.25 in [20])**.** *Assume that $\mathcal{D}$ and $\mathcal{E}$ are compatible Hamiltonian operators with $\mathcal{D}$ nondegenerate. If the differential functions Q_1, Q_2 and Q_3 satisfy*

$$\mathcal{E}Q_1 = \mathcal{D}Q_2, \qquad \mathcal{E}Q_2 = \mathcal{D}Q_3,$$

and there are local functionals F_1 and F_2 such that $Q_1 = \delta F_1$ and $Q_2 = \delta F_2$, then the Fréchet derivative D_{Q_3} is self-adjoint with respect to $(\cdot,\cdot)_{L^2}$.

Suppose $u \in X$ is a solution of (1.1) at a fixed time t. We let $m = u - u_{xx}$. Since $1 - D_x^2$ is an isomorphism between Sobolev spaces, $H^{n+2} \to H^n$, we see that u and m are in a one-to-one correspondence.

Henceforth, we let $\mathcal{D}$ and $\mathcal{E}$ be the operators

$$\mathcal{D} = -\left(D_x - D_x^3\right), \qquad \mathcal{E} = -(mD_x + D_x m).$$

$\mathcal{D}$ and $\mathcal{E}$ are compatible Hamiltonian operators (cf [6]). The Camassa–Holm equation is bi-Hamiltonian with the Hamiltonians

$$H_1[m] = \frac{1}{2}\int mu\,\mathrm{d}x, \qquad H_2[m] = \frac{1}{2}\int \left(u^3 + uu_x^2\right)\mathrm{d}x, \tag{2.2}$$

and the corresponding operators $\mathcal{E}, \mathcal{D}$. More precisely, equation (1.1) can be written as either

$$m_t = \mathcal{E}\delta H_1[m],$$

or

$$m_t = \mathcal{D}\delta H_2[m].$$

The integrals in (2.2) are to be interpreted in the following sense: for a function $m \in X$ the values of H_1 and H_2 at m are obtained by replacing u by $\left(1 - D_x^2\right)^{-1} m$ inside the integrals before computing them. We stress that H_1 and H_2 are viewed as functionals of m and not of u. The fact that they can be considered as functionals of both u and m can easily cause confusion if one is not careful. For example, the variational derivatives of the mappings $u \mapsto \frac{1}{2}\int mu\,\mathrm{d}x$ and $m \mapsto \frac{1}{2}\int mu\,\mathrm{d}x$ are different. Moreover, H_1 and H_2 are local as functionals of u, but not as functionals of m.

If two functionals F_0 and F_1 satisfy

$$\mathcal{E}\delta F_0 = \mathcal{D}\delta F_1,$$

then we say that F_0 *raises* to F_1 or that F_1 *lowers* to F_0: in symbols $F_0 \uparrow F_1$ or $F_1 \downarrow F_0$.

3. Conservation laws

The conservation laws for equation (1.1) are constructed as a sequence

$$\cdots \uparrow H_{-1} \uparrow H_0 \uparrow H_1 \uparrow H_2 \uparrow H_3 \uparrow \cdots.$$

Using the bi-Hamiltonian structure it is easy to prove that all functionals in this series are conserved under the flow of (1.1) (see [20] for a proof in the general case of a bi-Hamiltonian structure, and [6] for the special case of the Camassa–Holm equation).

The first few conservation laws in this sequence are

$$H_{-2}[m] = -\frac{1}{16}\int\left(\frac{4}{\sqrt{m}} + \frac{m_x^2}{m^{5/2}}\right)\mathrm{d}x, \qquad \delta H_{-2}[m] = \frac{1}{8m^{3/2}} + \frac{m_{xx}}{8m^{5/2}} - \frac{5m_x^2}{32m^{7/2}},$$

$$H_{-1}[m] = \int \sqrt{m}\,\mathrm{d}x, \qquad \delta H_{-1}[m] = \frac{1}{2\sqrt{m}},$$

$$H_0[m] = \int m\,\mathrm{d}x, \qquad \delta H_0[m] = 1,$$

$$H_1[m] = \frac{1}{2}\int mu\,\mathrm{d}x, \qquad \delta H_1[m] = u,$$

$$H_2[m] = \frac{1}{2}\int\left(u^3 + uu_x^2\right)\mathrm{d}x, \qquad \delta H_2[m] = \left(1 - D_x^2\right)^{-1}\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right).$$

We see that

$$H_{-2}[m] = -2\int m\delta H_{-2}[m]\,\mathrm{d}x, \qquad \delta H_{-2}[\lambda m] = \lambda^{-3/2}\delta H_{-2}[m],$$

$$H_{-1}[m] = 2\int m\delta H_{-1}[m]\,\mathrm{d}x, \qquad \delta H_{-1}[\lambda m] = \lambda^{-1/2}\delta H_{-1}[m],$$

$$H_0[m] = \int m\delta H_0[m]\,\mathrm{d}x, \qquad \delta H_0[\lambda m] = \delta H_0[m],$$

$$H_1[m] = \frac{1}{2}\int m\delta H_1[m]\,\mathrm{d}x, \qquad \delta H_1[\lambda m] = \lambda\delta H_1[m],$$

$$H_2[m] = \frac{1}{3}\int m\delta H_2[m]\,\mathrm{d}x, \qquad \delta H_2[\lambda m] = \lambda^2\delta H_2[m].$$

Therefore, it comes as no surprise when we prove that in general

$$\delta H_n[\lambda m] = \lambda^n\delta H_n[m], \qquad n = 0, 1, 2, \dots,$$

$$\delta H_{-n}[\lambda m] = \lambda^{1/2-n}\delta H_{-n}[m], \qquad n = 1, 2, \dots,$$

and

$$H_n[m] = \frac{1}{n+1}\int m\delta H_n[m]\,\mathrm{d}x, \qquad n = 0, 1, 2, \dots,$$

$$H_{-n}[m] = \frac{1}{3/2 - n}\int m\delta H_{-n}[m]\,\mathrm{d}x, \qquad n = 1, 2, \dots.$$

We will consider in turn the two cases of lowering and raising.

4. Lowering

We first need to describe the inverse of the Hamiltonian operator $\mathcal{E} = -(mD_x + D_x m)$.

Lemma 2. *We have*

$$\mathcal{E}^{-1} = -\frac{1}{2\sqrt{m}}D_x^{-1}\frac{1}{\sqrt{m}}.$$

Proof. Differentiation shows that for a function f

$$-\mathcal{E}\frac{1}{2\sqrt{m}}D_x^{-1}\left(\frac{f}{\sqrt{m}}\right) = (mD_x + D_x m)\frac{1}{2\sqrt{m}}D_x^{-1}\left(\frac{f}{\sqrt{m}}\right) = f. \qquad \square$$

The occurrence of D_x^{-1} in the expression for $\mathcal{E}^{-1}$ makes $\mathcal{E}^{-1}f$ defined only modulo functions of the form $\frac{c}{\sqrt{m}}$, $c \in \mathbb{R}$. For simplicity, whenever D_x^{-1} appears, we will choose the integration constant c to be zero.

Remark 1. The convention to take $c = 0$ does not make sense in the case of general functions: should one choose $D_x^{-1}(2\sin x\cos x) = \sin^2 x$ or $D_x^{-1}(2\sin x\cos x) = -\cos^2 x$?[1] However, consider the class $\mathcal{M}$ of differential functions with no constant term and with no explicit dependence on x. If $P[m] = P(m, m_x, m_{xx}, \ldots) \in \mathcal{M}$ is the total x-derivative of a differential function, then there is a unique $R[m] \in \mathcal{M}$ such that $D_x R[m] = P[m]$. Indeed, suppose $R_1[m], R_2[m] \in \mathcal{M}$ satisfy $D_x R_1[m] = D_x R_2[m] = P[m]$. Then $R_1[m] = R_2[m] + c$ for some constant c (see [20]). By assumption $R_1[m]$ contains no constant term. Moreover, since $R_2[m] \in \mathcal{M}$, c cannot be cancelled by terms in $R_2[m]$. We conclude that $c = 0$ so that $R_1[m] = R_2[m]$. This proves uniqueness.

To prove existence we note that any explicit x-dependence in a differential function $R[m]$ gives rise to a constant term or an explicit x-dependence in $D_x R[m]$. Therefore, if $D_x R[m] = P[m]$ for some $P[m] \in \mathcal{M}$, $R[m]$ cannot depend explicitly on x. Furthermore, discarding the possibly nonzero constant term of $R[m]$, we obtain a differential function in $\mathcal{M}$ with total x-derivative $P[m]$. This proves existence.

In the following D_x^{-1} will only be applied to differential functions $P[m]$ in the class $\mathcal{M}$. We define $D_x^{-1}P[m]$ to be the unique $R[m] \in \mathcal{M}$ satisfying $D_x R[m] = P[m]$.

To describe the domain of definition of $\mathcal{E}^{-1}$ we use the following lemma. Recall definition (2.1) of the Euler operator $\mathbf{E}$.

Lemma 3 (theorem 4.7 in [20])**.** *A differential function $L[m]$ satisfies the Euler–Lagrange equations $\mathbf{E}(L) = 0$ identically for all x, m, if and only if $L = D_x P$, for some differential function $P[m]$.*

If $F[m] = \int P[m]\,\mathrm{d}x \equiv 0$, then $\delta F[m] = \mathbf{E}(P) \equiv 0$. Thus, we get the following consequence of lemma 3.

Lemma 4. *If a differential function $P[m]$ satisfies*

$$\int P[m]\,\mathrm{d}x = 0,$$

for all m, then P is the total x-derivative $P = D_x R$ of some differential function $R[m]$.

Observe that lemma 4 says that $P[m] = D_x R[m]$ *for all m*. This is much deeper than the corresponding result for ordinary functions f of one real variable[2].

The proof that lowering is unobstructed and produces local conservation laws rests on the following lemma.

Lemma 5. *Let $Q_{-1} = \delta H_{-1} = \frac{1}{2\sqrt{m}}$. For each $n \geqslant 1$,*

$$R_{-n}[m] = \frac{1}{\sqrt{m}}\mathcal{D}(\mathcal{E}^{-1}\mathcal{D})^{n-1}Q_{-1}$$

[1] We thank the referees for this observation.

[2] If $\int f\,\mathrm{d}x = 0$, then clearly $g(x) = \int_{-\infty}^{x} f\,\mathrm{d}x$ satisfies $D_x g = f$ and the required boundary conditions. However, even if $\int P[m]\,\mathrm{d}x = 0$ for all m, it is far from obvious that there is a differential function $R[m]$ such that $P[m] = D_x R[m]$ for all m: the definition $R[m](x) = \int_{-\infty}^{x} P[m]\,\mathrm{d}x$ could very well necessitate a different R for different ms.

is the total x-derivative of a differential function. Hence, we may recursively define

$$Q_{-n-1} = \mathcal{E}^{-1}\mathcal{D}Q_{-n} = -\frac{1}{2\sqrt{m}}D_x^{-1}R_{-n}.$$

Moreover, the differential functions Q_{-n} satisfy

$$Q_{-n}[\lambda m] = \lambda^{1/2-n}Q_{-n}[m]. \tag{4.1}$$

Proof. Both $\mathcal{D}$ and $\mathcal{E}^{-1}$ are skew-adjoint operators, so that $(\mathcal{D}\mathcal{E}^{-1})^* = \mathcal{E}^{-1}\mathcal{D}$. We infer that

$$\int R_{-n}[m]\,\mathrm{d}x = \int \frac{1}{\sqrt{m}}\cdot\mathcal{D}(\mathcal{E}^{-1}\mathcal{D})^{n-1}Q_{-1}\,\mathrm{d}x = \frac{1}{2}\int\frac{1}{\sqrt{m}}\cdot(\mathcal{D}\mathcal{E}^{-1})^{n-1}\mathcal{D}\frac{1}{\sqrt{m}}\,\mathrm{d}x$$
$$= -\frac{1}{2}\int\mathcal{D}(\mathcal{E}^{-1}\mathcal{D})^{n-1}\left(\frac{1}{\sqrt{m}}\right)\cdot\frac{1}{\sqrt{m}}\,\mathrm{d}x = -\int R_{-n}[m]\,\mathrm{d}x.$$

Hence, $\int R_{-n}[m]\,\mathrm{d}x = 0$. Therefore, in view of lemma 4, if $R_{-n}[m]$ is a differential function, then

$$Q_{-n-1} = \frac{1}{\sqrt{m}}D_x^{-1}R_{-n},$$

is a differential function. But then also

$$R_{-n-1} = \frac{1}{\sqrt{m}}\mathcal{D}Q_{-n-1}$$

is a differential function. Since $R_{-1}[m]$ is clearly a differential function, the first half of the lemma follows by induction.

Now suppose

$$Q_{-j}[\lambda m] = \lambda^{1/2-j}Q_{-j}[m],$$

holds for $j = n$. Then,

$$Q_{-n-1}[\lambda m] = \mathcal{E}^{-1}\mathcal{D}Q_{-n}[\lambda m] = -\frac{1}{2\sqrt{\lambda m}}D_x^{-1}\frac{1}{\sqrt{\lambda m}}\mathcal{D}(\lambda^{1/2-n}Q_{-n}[m])$$
$$= -\lambda^{1/2-n-1}\frac{1}{2\sqrt{m}}D_x^{-1}\frac{1}{\sqrt{m}}\mathcal{D}Q_{-n}[m] = \lambda^{1/2-n-1}Q_{-n-1}[m].$$

Since

$$Q_{-1}[\lambda m] = \lambda^{-1/2}Q_{-1}[m],$$

an induction argument shows (4.1) for all $n \geqslant 1$. □

Now we know that there exist differential functions Q_{-n} such that

$$Q_{-1} = \delta H_{-1}, \qquad \mathcal{E}Q_{-n-1} = \mathcal{D}Q_{-n}, \qquad n \geqslant 1,$$

we may define local functionals H_{-n} by

$$H_{-n}[m] = \frac{1}{3/2-n}\int mQ_{-n}[m]\,\mathrm{d}x.$$

We compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}H_{-n}[m+\epsilon\eta] = \frac{1}{3/2-n}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\int(m+\epsilon\eta)Q_{-n}[m+\epsilon\eta]\,\mathrm{d}x$$
$$= \frac{1}{3/2-n}\left(\int\eta Q_{-n}[m]\,\mathrm{d}x + \int mD_{Q_{-n}}(\eta)\,\mathrm{d}x\right).$$

Since Q_{-1} and Q_{-2} are variational derivatives of functionals, an induction argument together with lemma 1 show that the Fréchet derivatives $D_{Q_{-n}}$ are self-adjoint for all $n \geqslant 1$. Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} H_{-n}[m+\epsilon\eta] = \frac{1}{3/2-n}\int \eta(Q_{-n}[m] + D_{Q_{-n}}(m))\,\mathrm{d}x. \tag{4.2}$$

Moreover, since $Q_{-n}[\lambda m] = \lambda^{1/2-n}Q_{-n}[m]$ by lemma 5, we obtain

$$D_{Q_{-n}}(m) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} Q_{-n}[m+\epsilon m] = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}(1+\epsilon)^{1/2-n}Q_{-n}[m] = (1/2-n)Q_{-n}[m].$$

Thus, (4.2) gives,

$$\delta H_{-n}[m] = \frac{1}{3/2-n}(Q_{-n}[m] + D_{Q_{-n}}(m)) = Q_{-n}[m].$$

This proves the following result.

Theorem 1. *Lowering of the H_ns is unobstructed for the Camassa–Holm equation, i.e., there are $H_{-1}, H_{-2}, \ldots$, such that $H_{-1} \downarrow H_{-2} \downarrow \ldots$. Moreover, $H_{-1}, H_{-2}, \ldots$ are local functionals and satisfy*

$$H_{-n}[m] = \frac{1}{3/2-n}\int m\delta H_{-n}[m]\,\mathrm{d}x, \qquad n = 1, 2, \ldots,$$

$$\delta H_{-n}[\lambda m] = \lambda^{1/2-n}\delta H_{-n}[m], \qquad n = 1, 2, \ldots.$$

5. Raising

Since $\left(1 - D_x^2\right)^{-1}$ is an isomorphism between Sobolev spaces, $H^k \to H^{k+2}$, we see that $\mathcal{D}^{-1}f = \left(1 - D_x^2\right)^{-1}D_x^{-1}f$ is well-defined whenever f is an x-derivative.

Lemma 6. *Let $Q_0 = \delta H_0 = 1$. For each $n \geqslant 0$, $\mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0$ is an x-derivative. Hence, we may recursively define*

$$Q_{n+1} = \mathcal{D}^{-1}\mathcal{E}Q_n.$$

Moreover,

$$Q_n[\lambda m] = \lambda^n Q_n[m], \qquad n = 0, 1, 2, \ldots. \tag{5.1}$$

Proof. First observe that $\mathcal{E}$ and $\mathcal{D}^{-1}$ are skew-symmetric operators and that $(\mathcal{E}\mathcal{D}^{-1})^* = \mathcal{D}^{-1}\mathcal{E}$. This yields

$$\int \mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0\,\mathrm{d}x = \int (\mathcal{E}\mathcal{D}^{-1})^n\mathcal{E}(1)\cdot 1\,\mathrm{d}x = -\int 1\cdot\mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n(1)\cdot 1\,\mathrm{d}x$$
$$= -\int \mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0\,\mathrm{d}x.$$

We conclude that

$$\int \mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0\,\mathrm{d}x = 0,$$

so that

$$f(x) = \int^x \mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0\,\mathrm{d}x,$$

satisfies the right boundary conditions and $D_x f = \mathcal{E}(\mathcal{D}^{-1}\mathcal{E})^n Q_0$.

To show (5.1) we proceed by induction on n. Assume $Q_n[\lambda m] = \lambda^n Q_n[m]$ holds. Then,

$$\begin{aligned} Q_{n+1}[\lambda m] &= \mathcal{D}^{-1}\mathcal{E} Q_n[\lambda m] = \left(1 - D_x^2\right)^{-1} D_x^{-1}(\lambda m D_x + D_x \lambda m)(\lambda^n Q_n[m]) \\ &= \lambda^{n+1}\left(1 - D_x^2\right)^{-1} D_x^{-1}(m D_x + D_x m) Q_n[m] = \lambda^{n+1} Q_{n+1}[m]. \end{aligned}$$

Since (5.1) obviously holds for $n = 0$, this shows (5.1) for all $n \geqslant 0$ and completes the proof. □

Lemma 6 shows existence of Q_n such that

$$Q_0 = \delta H_0, \qquad \mathcal{E} Q_n = \mathcal{D} Q_{n+1}, \qquad n \geqslant 0,$$

so that we may define

$$H_n[m] = \frac{1}{n+1} \int m Q_n[m]\, \mathrm{d}x, \qquad n = 0, 1, 2, \ldots. \tag{5.2}$$

If the Fréchet derivative D_{Q_n} were self-adjoint we would proceed just like in the case of the lowering to deduce that $\delta H_n[m] = Q_n[m]$. This would show that raising is unobstructed. However, since the Q_ns are non-local expressions for $n \geqslant 3$, we cannot immediately apply lemma 1 or the theory from [20] to infer self-adjointness. This technical problem is usually ignored.

At any rate, if there is a functional H_n with variational derivative $\delta H_n = Q_n$, then $D_{Q_n} = D_{\delta H_n}$ is self-adjoint so that H_n is given by (5.2). We can therefore state the following theorem.

Theorem 2. *Whenever the functionals H_n exist, it holds that*

$$H_n[\lambda m] = \lambda^{n+1} H_n[m], \qquad n = 0, 1, 2, \ldots,$$

and

$$H_n[m] = \frac{1}{n+1} \int m \delta H_n[m]\, \mathrm{d}x, \qquad n = 0, 1, 2, \ldots.$$

□

6. Algorithm

In this section, we use the previously derived identities to obtain a constructive algorithm for computing the H_ns.

6.1. Lowering

In view of theorem 1, we have

$$H_{-n}[m] = \frac{1}{3/2 - n} \int m \delta H_{-n}[m]\, \mathrm{d}x, \qquad n = 1, 2, \ldots. \tag{6.1}$$

Using that

$$\delta H_{-n}[m] = \mathcal{E}^{-1}\mathcal{D}\delta H_{-n+1}[m] = -\frac{1}{2\sqrt{m}} D_x^{-1}\left(\frac{\mathcal{D}\delta H_{-n+1}[m]}{\sqrt{m}}\right),$$

we obtain

$$\delta H_{-n}[m] = \frac{1}{2\sqrt{m}} \int_0^x \frac{\left(D_x - D_x^3\right)\delta H_{-n+1}[m]}{\sqrt{m}}\, \mathrm{d}\xi. \tag{6.2}$$

Formula (6.2) provides an explicit recursive algorithm for the computation of the δH_{-n}s. Once $\delta H_{-n}[m]$ is known, $H_{-n}[m]$ is obtained from (6.1). To exemplify the approach we construct the conservation laws H_{-3} and H_{-4}.

We first use the integration by parts formula $D_x^{-1}(uv_x) = uv - D_x^{-1}(u_x v)$ twice in the expression

$$\delta H_{-3}[m] = \frac{1}{2\sqrt{m}} D_x^{-1}\left(\frac{(D_x - D_x^3)\delta H_{-2}[m]}{\sqrt{m}}\right)$$

to get

$$\delta H_{-3}[m] = \frac{1}{2\sqrt{m}}\left[\frac{1}{\sqrt{m}}(1 - D_x^2)\delta H_{-2}[m] + \left(\frac{1}{\sqrt{m}}\right)_x (\delta H_{-2}[m])_x - D_x^{-1}\left(\left(\frac{1}{\sqrt{m}}\right)_x \delta H_{-2}[m] + \left(\frac{1}{\sqrt{m}}\right)_{xx}(\delta H_{-2}[m])_x\right)\right].$$

Now since

$$\delta H_{-2}[m] = \frac{1}{8m^{3/2}} + \frac{m_{xx}}{8m^{5/2}} - \frac{5m_x^2}{32m^{7/2}},$$

a long but straightforward[3] computation yields

$$\delta H_{-3}[m] = \frac{3}{64m^{5/2}} + \frac{5m_{xx}}{32m^{7/2}} - \frac{35m_x^2}{128m^{9/2}} - \frac{m_{xxxx}}{16m^{7/2}} + \frac{7m_x m_{xxx}}{16m^{9/2}} - \frac{231m_x^2 m_{xx}}{128m^{11/2}} + \frac{21m_{xx}^2}{64m^{9/2}} + \frac{1155m_x^4}{1024m^{13/2}}.$$

From (6.1) we get

$$H_{-3}[m] = -\frac{2}{3}\int m\delta H_{-3}[m]\,\mathrm{d}x = -\int\left(\frac{1}{32m^{3/2}} + \frac{5m_{xx}}{48m^{5/2}} - \frac{35m_x^2}{192m^{7/2}} - \frac{m_{xxxx}}{24m^{5/2}} + \frac{7m_x m_{xxx}}{24m^{7/2}} - \frac{77m_x^2 m_{xx}}{64m^{9/2}} + \frac{7m_{xx}^2}{32m^{7/2}} + \frac{385m_x^4}{512m^{11/2}}\right)\mathrm{d}x.$$

After some integrations by parts, we arrive at

$$H_{-3}[m] = -\int\left(\frac{1}{32m^{3/2}} + \frac{5m_x^2}{64m^{7/2}} + \frac{m_{xx}^2}{32m^{7/2}} + \frac{35m_x^4}{512m^{11/2}}\right)\mathrm{d}x.$$

Repeating the same steps again with $\delta H_{-2}[m]$ replaced by $\delta H_{-3}[m]$, we obtain

$$\begin{aligned}\delta H_{-4}[m] = {} & \frac{5}{256m^{7/2}} + \frac{1419m_x m_{xx} m_{xxx}}{128m^{13/2}} + \frac{m_{xxxxxx}}{32m^{9/2}} - \frac{7m_{xxxx}}{64m^{9/2}} - \frac{425\,425m_x^6}{16\,384m^{19/2}} \\ & + \frac{15\,015m_x^4}{4096m^{15/2}} + \frac{35m_{xx}}{256m^{9/2}} - \frac{315m_x^2}{1024m^{11/2}} + \frac{189m_{xx}^2}{256m^{11/2}} + \frac{671m_{xx}^3}{256m^{13/2}} - \frac{69m_{xxx}^2}{128m^{11/2}} \\ & - \frac{35\,607m_x^2 m_{xx}^2}{1024m^{15/2}} + \frac{255\,255m_x^4 m_{xx}}{4096m^{17/2}} - \frac{2145m_x^3 m_{xxx}}{128m^{15/2}} + \frac{825m_x^2 m_{xxxx}}{256m^{13/2}} \\ & - \frac{2541m_x^2 m_{xx}}{512m^{13/2}} - \frac{57m_{xx} m_{xxxx}}{64m^{11/2}} + \frac{63m_x m_{xxx}}{64m^{11/2}} - \frac{27m_x m_{xxxxx}}{64m^{11/2}}.\end{aligned}$$

[3] To find the primitive function of

$$R[m] = \left(\frac{1}{\sqrt{m}}\right)_x \delta H_{-2}[m] + \left(\frac{1}{\sqrt{m}}\right)_{xx} (\delta H_{-2}[m])_x$$

we first compensate for the highest order derivatives and then progressively work our way down. Since $R[m]$ is the total x-derivative of a differential function by theorem 1, this procedure will stop and produce the right result.

Hence, (6.1) gives after integrating by parts

$$H_{-4}[m] = -\int\left(\frac{1}{128m^{5/2}} + \frac{35m_x^2}{512m^{9/2}} + \frac{7m_{xx}^2}{128m^{9/2}} + \frac{m_{xxx}^2}{64m^{9/2}} - \frac{105m_x^2 m_{xx}}{1024m^{11/2}} + \frac{m_{xx}^3}{256m^{11/2}} + \frac{21m_x m_{xx} m_{xxx}}{128m^{11/2}} + \frac{5005m_x^4 m_{xx}}{12288m^{15/2}}\right)\mathrm{d}x.$$

These computations, although very laborious, could clearly be continued to produce explicit expressions for H_{-5}, H_{-6}, etc.

6.2. Raising

By theorem 2 we have

$$H_n[m] = \frac{1}{n+1}\int m\delta H_n[m]\,\mathrm{d}x, \qquad n = 0, 1, 2, \ldots. \tag{6.3}$$

Since

$$\delta H_n[m] = \mathcal{D}^{-1}\mathcal{E}\delta H_{n-1}[m] = \left(1 - D_x^2\right)^{-1} D_x^{-1}(mD_x + D_x m)\delta H_{n-1}[m],$$

we find that

$$\delta H_n[m] = \left(1 - D_x^2\right)^{-1}\int_0^x (2m(\delta H_{n-1}[m])_x + m_x\delta H_{n-1}[m])\,\mathrm{d}\xi.$$

Moreover,

$$\left(1 - D_x^2\right)^{-1} f(x) = (p * f)(x) = \int p(x-y) f(y)\,\mathrm{d}y,$$

where, if we are on the line,

$$p(x) = \tfrac{1}{2}\,\mathrm{e}^{-|x|}, \qquad x \in \mathbb{R},$$

or

$$p(x) = \frac{\cosh(1/2 - x)}{2\sinh 1/2}, \qquad x \in [0, 1),$$

for the period one case. Therefore,

$$\delta H_n[m] = \int p(x-y)\int_0^y (2m(\delta H_{n-1}[m])_x + m_x\delta H_{n-1}[m])\,\mathrm{d}\xi\,\mathrm{d}y. \tag{6.4}$$

This formula gives a recursive scheme to compute the δH_ns. Since the functionals become non-local for $n \geqslant 3$, the algorithm does not produce as explicit formulae as in the case of lowering. Nevertheless, starting with for example $\delta H_2[m] = \left(1 - D_x^2\right)^{-1}\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)$, the scheme can be implemented numerically to yield δH_3, δH_4, etc. Once $\delta H_n[m]$ is known, the value of $H_n[m]$ is obtained from (6.3). Note that this last step assumes the existence of the functionals $H_n[m]$, $n \geqslant 4$, even though this was never rigorously proved (cf the remark at the end of section 5).

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